Journal of Approximation Theory **119**, 86–109 (2002)
doi:10.1006/jath.2002.3721

The Complex Spline Approximation of Singular Integral Operators over an Open Arc

Yong Jia Xu

Department of Statistics, Guangdong Business College, Guangzhou, Guangdong 510320, People's Republic of China
E-mail: yjxu@gdgz.cnuninet.com

Communicated by Wolfgang Dahmen

Received August 29, 2001; accepted in revised form June 26, 2002

This paper mainly considers the smooth complex spline approximation of Cauchy-type integral operators over an open arc. First, the smoothness of the operators is investigated, then some properties of complex splines are discussed, and finally the error estimates of the approximation are given. © 2002 Elsevier Science (USA)

1. INTRODUCTION

The approximation for Cauchy-type singular integrals and the numerical solution of the corresponding equations have been studied in many papers, and a series of important results have been obtained [3, 7, 8, 13]. However, the rates of convergence for the approximation are not very satisfactory. In fact, the best rates have not been revealed. The reason may be that, as mappings, the smoothness of the integrals is not investigated thoroughly. For example, paper [7] discusses the following operator:

$$\hat{A}(\varphi)(t) \equiv a(t)w_2(t)\varphi(t) - \frac{b(t)}{\pi}\int_{-1}^{1}\frac{w_2(\tau)\varphi(\tau)}{\tau - t}\,\mathrm{d}\tau, \qquad t \in [-1, 1], \tag{1.1}$$

where $a \in H[-1, 1]$, b is a polynomial and w_2 is a weight function constructed from a and b (for details, see [7] or [6]), and proves that if $w_2 \in H^{\gamma}[-1, 1]$ and $\varphi \in C^{m,\mu}[-1, 1]$ for $0<\gamma, \mu<1$, then $\hat{A}\varphi \in H^{\lambda}[-1, 1]$ where $\lambda = \min(\gamma, \mu)$. This means that the smoothness of $\hat{A}\varphi$ depends on w_2. But in fact, as an operator, $\hat{A}$ is sufficiently smooth and its smoothness is not affected by w_2. This can be illustrated when $a(t) \equiv 0$ and $b(t) \equiv -1$ in (1.1). In this case, $\hat{A}$ becomes

$$T(\varphi)(t) \equiv \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\varphi(\tau)}{\tau - t}\,\mathrm{d}\tau, \qquad t \in [-1, 1] \tag{1.2}$$

0021-9045/02 $35.00

and it is bounded on $C^{m,\mu}[-1,1]$ because we can easily convert it to a periodic singular integral or a Cauchy-type integral on a unit circle (cf. [15, 18]).

Usually, the Cauchy-type singular integrals are of the form

$$T_\Gamma(\varphi)(t) \equiv \frac{1}{\pi}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,\mathrm{d}\tau, \qquad t \in \Gamma, \tag{1.3}$$

where Γ is a smooth or piecewise smooth oriented curve. If Γ is closed, as a mapping, T_Γ is bounded on $C^{m,\mu}(\Gamma)$, and if Γ is an axis, T_Γ has the similar property [14, 15]. However, if Γ is an open arc, the property disappears. Instead, we consider its "weighted type". This is also a natural approach to the solution of singular integral equations over an open arc.

Generally speaking, by a simple transformation, our problems on an open arc can be converted to those on an interval, and then the problems may be solved more conveniently. But this will bring about at least two problems: the first is that the kernels of those integrals may become more complicated and it will then be more difficult for some treatments, such as numerical evaluations; the second is that a smooth function on the arc, as well as the related smooth operators, may become less or even not smooth on the interval if the arc is not perfectly smooth, and it will then impair the rates of its approximation. On the other hand, it is known that complex splines have a lot of advantages over other functions such as polynomials in approximation, especially in the approximation for those functions defined on arbitrary curves. So it will be more meaningful to discuss the singular integral operators and their complex spline approximation directly on arcs.

In this paper, our discussion focuses on the Cauchy singular integral operators of the form

$$A_w(\varphi)(t) \equiv a(t)w(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{w(\tau)\varphi(\tau)}{\tau - t}\,\mathrm{d}\tau, \qquad t \in \Gamma, \tag{1.4}$$

which are derived from the theory of singular integral equations. Here we assume $a, b \in H(\Gamma)$ satisfying

$$a^2(t) - b^2(t) \neq 0, \qquad t \in \Gamma$$

and Γ is a smooth or piecewise smooth oriented open arc from α to β. The function w is constructed as follows (cf. [5, 6, 12]).

Let $G(t) = \frac{a(t)+b(t)}{a(t)-b(t)}$, and choose a continuous branch of $G(t)$ such that

$$0 \leqslant \theta(t) < 1, \tag{1.5}$$

where $\theta(t) = \frac{\arg G(t)}{2\pi}$. Let $[y]$ denote the integral part of the real number y and

$$\kappa = -[-\theta(\beta)]. \tag{1.6}$$

Define the canonical function [12, 15]

$$X(z) = (\beta - z)^{-\kappa} \exp\left[\frac{1}{2\pi i}\int_\Gamma \frac{\ln G(\tau)}{\tau - z}\,\mathrm{d}\tau\right], \qquad z \notin \Gamma \tag{1.7}$$

and now let

$$w(t) = [(a(t) - b(t))X^+(t)]^{-1}, \qquad t \in \Gamma. \tag{1.8}$$

It is easy to verify that $w(t) \in H(\Gamma)$ and $w(t) \neq 0, t \in \Gamma \backslash \{\alpha, \beta\}$, which is similar to a weight function.

From the above construction, we know the smoothness of the function w is poor. However, we will show in this paper that the operator A_w is very smooth. Furthermore, if $w(\alpha) = w(\beta) = 0$ and $b \in C^{m,\mu}(\Gamma)$, A_w is a bounded operator on $C^{m,\mu}(\Gamma)$. Besides, we make a further study of complex interpolating splines and obtain some interesting properties. And then, there follow the results of the complex spline approximation for the operator A_w. All these appear to be very attractive.

Throughout the paper, we assume that: $0 < \mu < 1$, m is a nonnegative integer, $\Gamma = \overset{\frown}{\alpha\beta}$, c_0 is a constant such that the arc length $|\overset{\frown}{t_1 t_2}| \leqslant c_0 |t_1 - t_2|$ for all $t_1, t_2 \in \Gamma$, $C(\Gamma)$ and $C^m(\Gamma)$ denote the spaces of continuous and m-times continuously differentiable complex-valued functions on Γ, respectively, $\|\psi\| \equiv \max_{t \in \Gamma} |\psi(t)|$, $\|\psi\|_{c^m} \equiv \sum_{k=0}^m \|\psi^{(k)}\|$, the modulus of continuity for $\psi \in C(\Gamma)$ is denoted by $\omega(\psi, x)$, $M_\mu(\psi) \equiv \sup_{0 < x \leqslant 1} \frac{\omega(\psi, x)}{x^\mu}$, $C^{m,\mu}(\Gamma) \equiv \{\psi \in C^m(\Gamma) : M_\mu(\psi^{(m)}) < \infty\}$, $H^\mu(\Gamma) \equiv C^{0,\mu}(\Gamma)$, $H(\Gamma) \equiv \bigcup_{0 < \mu < 1} H^\mu(\Gamma)$, $\|\psi\|_{H^\mu} \equiv \|\psi\| + M_\mu(\psi)$, $\|\psi\|_{C^{m,\mu}} \equiv \|\psi\|_{C^m} + M_\mu(\psi^{(m)})$, and c is an absolute positive constant taking different value in different place.

The rest of the paper is organized as follows. In the next section, we show the smoothness of the operator A_w. Some properties of complex splines are discussed in Section 3 and the results of spline approximation for A_w are given in Section 4. In Section 5 we illustrate an application for the approximation and remark on the case $\mu = 1$, constant c and some other applications. The proof of Lemma 2.2 is longer and thus is put in the final section.

2. SMOOTHNESS OF SINGULAR INTEGRAL OPERATORS

In this section, we start our discussion for the smoothness of operator A_w from the introduction of some operators and the proofs of two lemmas.

Let $u \in C(\Gamma)$. Define the operator $S_{u,m}$ as

$$S_{u,m}(f)(t) = m! \int_\Gamma u\tau \frac{f(\tau) - T_m(f)(\tau, t)}{(\tau - t)^{m+1}}\,\mathrm{d}\tau, \qquad f \in C^{m,\mu}(\Gamma),$$

where $T_m(f)(\tau,t)=f(t)+f'(t)(\tau-t)+\cdots+\frac{f^{(m)}(t)}{m!}(\tau-t)^m$ and $T_0(f)(\tau,t)=f(t)$. If $m=0$, $S_{u,m}(f)$ is written as $S_u(f)$.

Let $\Psi_k(\tau,t)=k!u(\tau)\frac{f(\tau)-T_k(f)(\tau,t)}{(\tau-t)^{k+1}}$, $f\in C^{m,\mu}(\Gamma)$. It is easy to verify that

$$\frac{\partial}{\partial t}\Psi_{k-1}(\tau,t)=\Psi_k(\tau,t) \tag{2.1}$$

and

$$\sup_{t\in\Gamma}\int_\Gamma|\Psi_k(\tau,t)||d\tau|<\infty, \tag{2.2}$$

and using dominant convergence theorem, we obtain

$$\frac{d^k}{dt^k}S_u(f)(t)=S_{u,k}(f)(t),\qquad t\in\Gamma \tag{2.3}$$

for $k=1,2,\ldots,m$.

If $k<m$, then

$$\begin{aligned}|f(\tau)-T_k(f)(\tau,t)|&=\left|\frac{1}{k!}\int_t^\tau(\zeta-\tau)^kf^{(k+1)}(\zeta)\,\mathrm{d}\zeta\right|\\&\leqslant c\frac{1}{k!}\|f^{(k+1)}\|\,|\tau-t|^{k+1}\end{aligned} \tag{2.4}$$

and if $k=m$,

$$\begin{aligned}&|f(\tau)-T_m(f)(\tau,t)|\\&\quad=\left|\frac{1}{(m-1)!}\int_t^\tau(\zeta-\tau)^{m-1}[f^{(m)}(\zeta)-f^{(m)}(t)]\,\mathrm{d}\zeta\right|\\&\quad\leqslant\frac{c}{m!}\omega(f^{(m)},|\tau-t|)|\tau-t|^m\end{aligned} \tag{2.5}$$

for $\tau,t\in\Gamma$, therefore

$$|S_{u,k}(f)(t)|\leqslant c\int_\Gamma|u(\tau)|\,\|f^{(k+1)}\|\,|d\tau|\leqslant c'\|u\|\,\|f^{(k+1)}\|,\qquad t\in\Gamma \tag{2.6}$$

for $k=1,2,\ldots,m-1$, and

$$|S_{u,m}(f)(t)|\leqslant c\|u\|\int_0^1\frac{\omega(f^{(m)},y)}{y}\,\mathrm{d}y,\qquad t\in\Gamma. \tag{2.7}$$

Thus, we have

Lemma 2.1. *For the operator* S_u, *there hold* (2.3) *and the following estimates*:

$$\left\|\frac{d^k}{dt^k}S_u(f)\right\| \leqslant c\|u\|\,\|f^{(k+1)}\|, \qquad k=0,1,\ldots,m-1, \tag{2.8}$$

if $f \in C^m$ *and*

$$\left\|\frac{d^m}{dt^m}S_u(f)\right\| \leqslant c\|u\| \int_0^1 \frac{\omega(f^{(m)},y)}{y}\,dy, \tag{2.9}$$

if $f^{(m)}$ *satisfies Dini condition, i.e.*, $\int_0^1 \frac{\omega(f^{(m)},x)}{x}dx < \infty$.

We also have

Lemma 2.2. *There is a constant* $c>0$ *depending on* m, μ *and* Γ *such that*

$$\omega\left(\frac{d^m}{dt^m}S_u(f), x\right) \leqslant c\|u\| M_\mu(f^{(m)}) x^\mu (1+|\ln x|) \tag{2.10}$$

or

$$\omega\left(\frac{d^m}{dt^m}S_u(f), x\right) \leqslant c\left(\|u\| + \int_0^1 \frac{\omega(u,y)}{y}\,\mathrm{d}y\right) M_\mu(f^{(m)}) x^\mu, \tag{2.11}$$

if u *satisfies Dini condition and* $u(\alpha)=u(\beta)=0$, *where* $f \in C^{m,\mu}(\Gamma)$ *and* $x>0$.

Since the proof is longer, we put it in the last section.

From Lemmas 2.1 and 2.2, it follows that

$$\|S_u(f)\|_{C^{m,\nu}} \leqslant cN(u)\|f\|_{C^{m,\mu}}, \tag{2.12}$$

where $0<\nu<\mu$ and $N(u)=\|u\|$ or $0<\nu\leqslant\mu$ and $N(u)=\|u\|+\int_0^1 \frac{\omega(u,x)}{x}\,\mathrm{d}x$ if u satisfies Dini condition and $u(\alpha)=u(\beta)=0$.

We rewrite the operator A_w as

$$A_w(\varphi)(t) = A_w^0(\varphi)(t) - \frac{1}{\pi i}\int_\Gamma w(\tau)\frac{b(\tau)-b(t)}{\tau-t}\varphi(\tau)\,\mathrm{d}\tau, \qquad t\in\Gamma, \tag{2.13}$$

where

$$A_w^0(\varphi)(t) \equiv a(t)w(t)\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{w(\tau)b(\tau)\varphi(\tau)}{\tau-t}\,\mathrm{d}\tau, \qquad t\in\Gamma \tag{2.14}$$

and let

$$K(\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma} w(\tau)\frac{b(\tau)-b(t)}{\tau - t}\varphi(\tau)\,\mathrm{d}\tau, \quad t \in \Gamma$$

or

$$K(\varphi)(t) = \frac{1}{\pi i} S_{w\varphi}(b). \tag{2.15}$$

Thus,

$$A_w = A_w^0 - K. \tag{2.16}$$

THEOREM 2.3. *There exists a constant* $c > 0$ *such that*

$$\left\|\frac{d^k}{dt^k}A_w^0(\varphi)\right\| \leqslant c\|\varphi\|_{C^{k+1}}, \qquad k = 0, 1, \ldots, m-1 \tag{2.17}$$

for $\varphi \in C^m$ *and*

$$\|A_w^0(\varphi)\|_{C^{m,\nu}} \leqslant c\|\varphi\|_{C^{m,\mu}} \tag{2.18}$$

for $\varphi \in C^{m,\mu}(\Gamma)$, *where* $0 < \nu < \mu$. *If* $b(\alpha)w(\alpha) = b(\beta)w(\beta) = 0$, *then* (2.18) *is valid for* $\nu = \mu$.

Proof. A_w^0 can be written as

$$A_w^0(\varphi)(t) = A_w^0(1)(t)\varphi(t) + \frac{1}{\pi i}\int_{\Gamma}\frac{w(\tau)b(\tau)[\varphi(\tau)-\varphi(t)]}{\tau - t}\,d\tau, \qquad t \in \Gamma$$

or

$$A_w^0(\varphi)(t) = p(t)\varphi(t) + \frac{1}{\pi i}S_{wb}(\varphi)(t), \tag{2.19}$$

where $p(t) = A_w^0(1)(t)$. But

$$p(t) = p.p.\left(\frac{1}{X(z)}, t\right) \tag{2.20}$$

is a κ-degree polynomial ($p(t) \equiv 0$ if $\kappa < 0$) (cf. [16]), and it is obvious that

$$\|(p\varphi)^{(k)}\| \leqslant c\|\varphi\|_{C^k}, \qquad k = 0, 1, \ldots, m \tag{2.21}$$

for $\varphi \in C^m(\Gamma)$ and

$$M_\nu((p\varphi)^{(m)}) \leqslant c\|\varphi\|_{C^{m,\mu}}, \qquad 0 \leqslant \nu \leqslant \mu \tag{2.22}$$

or

$$||p\varphi||_{C^{m,\nu}} \leqslant c||\varphi||_{C^{m,\mu}} \tag{2.23}$$

for $\varphi \in C^{m,\mu}(\Gamma)$, where the constant c depends on $a(t)$, $b(t)$ and m. On the other hand, wb satisfies Dini condition, so that there hold (2.8)–(2.10) for S_{wb} with $u = wb$ and there also holds (2.11) if $w(\alpha)b(\alpha) = w(\beta)b(\beta) = 0$. Thus we obtain (2.17) and (2.18) from (2.19). The proof is complete. ∎

Let $b \in C^{m,\mu}(\Gamma)$. From Lemmas 2.1 and 2.2,

$$\left|\left|\frac{d^k}{dt^k} K(\varphi)\right|\right| \leqslant c'||w\varphi||\,||b^{(k+1)}|| \leqslant c||\varphi||, \qquad k = 0, 1, \ldots, m-1 \tag{2.24}$$

and

$$||K(\varphi)||_{C^{m,\nu}} \leqslant c||w\varphi||\,||b||_{C^{m,\mu}} \leqslant c||w||\,||b||_{C^{m,\mu}}||\varphi||, \qquad 0 < \nu < \mu \tag{2.25}$$

for $\varphi \in C(\Gamma)$, and if $w(\alpha) = w(\beta) = 0$, then

$$\begin{aligned} ||K(\varphi)||_{C^{m,\nu}} &\leqslant c'\left(||w\varphi|| + \int_0^1 \frac{w(w\varphi, y)}{y}\,\mathrm{d}y\right)||b||_{C^{m,\mu}} \\ &\leqslant c\left(||\varphi|| + \int_0^1 \frac{\omega(\varphi, y)}{y}\,\mathrm{d}y\right) \end{aligned} \tag{2.26}$$

for $\varphi \in H^{\mu}(\Gamma)$, where we have used the fact that

$$\omega(w\varphi, y) \leqslant ||w||\omega(\varphi, y) + ||\varphi||\omega(w, y).$$

Thus, we obtain

Theorem 2.4. *Let $b \in C^{m,\mu}(\Gamma)$. Then there is a constant $c > 0$ such that*

$$\left|\left|\frac{d^k}{dt^k} A_w(\varphi)\right|\right| \leqslant c||\varphi||_{C^{k+1}}, \qquad k = 0, 1, \ldots, m-1 \tag{2.27}$$

for $\varphi \in C^m(\Gamma)$ and

$$||A_w(\varphi)||_{C^{m,\nu}} \leqslant c||\varphi||_{C^{m,\mu}} \tag{2.28}$$

for $\varphi \in C^{m,\mu}(\Gamma)$, where $0 < \nu < \mu$. If $w(\alpha) = w(\beta) = 0$, then (2.28) *is valid for $\nu = \mu$.*

Example 2.1. Consider the operator $\hat{A}$ in (1.1) of Section 1. Since a, b are real functions, we have $w_2(\pm 1)b(\pm 1) = 0$ ([16]). And b is a polynomial, so that $\hat{A}$ is bounded on $C^{m,\mu}(\Gamma)$ according to Theorem 2.4. An interesting result is that the operator

$$\hat{A}^0(\varphi)(t) \equiv a(t)w_2(t)\varphi(t) - \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(\tau)b(\tau)\varphi(\tau)}{\tau - t}d\tau, \qquad t \in [-1, 1] \qquad (2.29)$$

is bounded on $C^{m,\mu}(\Gamma)$ where we only require $a, b \in H[-1, 1]$. It appears that its smoothness does not depend on a and b.

3. SOME PROPERTIES OF COMPLEX SPLINES

Complex splines have many good properties (for details, see [1, 2, 4, 10, 11]). In order to discuss the approximation of singular integral operators, we need to have a further investigation for the splines. Here we only refer to the linear and cubic interpolating splines.

Let

$$\Delta : \alpha = t_0 \prec t_1 \prec \cdots \prec t_N = \beta$$

be a partition of Γ, $\Gamma_j = \overset{\frown}{t_{j-1}t_j}$, $h = \max_{1 \leqslant j \leqslant N} |\Delta t_j|$, and $f_j^{(r)} = f^{(r)}(t_j)$, where $t_{j-1} \prec t_j$ means that t_{j-1} precedes t_j, $\Delta t_j = t_j - t_{j-1}$ and $f \in C^m(\Gamma)$. We denote the linear and cubic interpolating splines of f by $s_1(f)$ or s_1 and $s_3(f)$ or s_3, respectively.

For linear interpolating spline, we have

$$s_1(t) = \frac{t_j - t}{\Delta t_j} f_{j-1} + \frac{t - t_{j-1}}{\Delta t_j} f_j, \qquad t \in \Gamma_j, \quad j = 1, 2, \ldots, N$$

and

$$\|s_1(f) - f\| \leqslant c_0 \omega(f, h). \qquad (3.1)$$

Let $t, t' \in \Gamma$, $t' \prec t$ and $|t - t'| > 0$. Then there are k and j, $0 \leqslant k \leqslant j \leqslant N$, such that $t \in \Gamma_j$ and $t' \in \Gamma_k$. If $k < j$, we have

$$|s_1(t) - s_1(t')| \leqslant |s_1(t) - f_{j-1}| + |f_{j-1} - f_k| + |f_k - s_1(t')|.$$

According to the property of modulus of continuity,

$$\begin{aligned} |f_{j-1} - f_k| &\leqslant \omega(f, |t_{j-1} - t_k|) \\ &\leqslant \left(1 + \frac{|t_{j-1} - t_k|}{|t - t'|}\right)\omega(f, |t - t'|), \end{aligned}$$

$$\begin{aligned}|s_1(t) - f_{j-1}| &\leqslant \frac{|t-t_{j-1}|}{|\Delta t_j|}\omega(f, |\Delta t_j|) \\ &\leqslant \frac{|t-t_{j-1}|}{|\Delta t_j|}\left(1+\frac{|\Delta t_j|}{|t-t'|}\right)\omega(f, |t-t'|) \\ &\leqslant \left(c_0 + \frac{|t-t_{j-1}|}{|t-t'|}\right)\omega(f, |t-t'|)\end{aligned}$$

and similarly,

$$|f_k - s_1(t')| \leqslant \left(c_0 + \frac{|t_k - t'|}{|t-t'|}\right)\omega(f, |t-t'|).$$

So we have

$$\begin{aligned}|s_1(t) - s_1(t')| &\leqslant \left(1 + 2c_0 + \frac{|t-t_{j-1}| + |t_{j-1} - t_k| + |t_k - t'|}{|t-t'|}\right)\omega(f, |t-t'|) \\ &\leqslant (1+3c_0)\omega(f, |t-t'|).\end{aligned}$$

If $k = j$, then

$$\begin{aligned}|s_1(t) - s_1(t')| &\leqslant \frac{|t-t'|}{|\Delta t_j|}\omega(f, |\Delta t_j|) \\ &\leqslant \frac{|t-t'|}{|\Delta t_j|}\left(1+\frac{|\Delta t_j|}{|t-t'|}\right)\omega(f, |t-t'|) \\ &\leqslant (1+c_0)\omega(f, |t-t'|).\end{aligned}$$

Thus, we have proved that

$$\omega(s_1, x) \leqslant (1+3c_0)\omega(f, x), \qquad x \geqslant 0. \tag{3.2}$$

From (3.1) and (3.2), we conclude that

$$\omega(s_1 - f, x) \leqslant c\omega(f, x^\gamma h^{1-\gamma}), \qquad x \geqslant 0, \tag{3.3}$$

where $c = 2 + 3c_0,\ 0 \leqslant \gamma \leqslant 1$. We show the conclusion as follows.

If $0 \leqslant x \leqslant h$, then $x \leqslant x^\gamma h^{1-\gamma}$ and

$$\begin{aligned}\omega(s_1 - f, x) &\leqslant \omega(s_1, x) + \omega(f, x) \\ &\leqslant (2+3c_0)\omega(f, x) \\ &\leqslant c\omega(f, x^\gamma h^{1-\gamma})\end{aligned}$$

by (3.2); if $x > h$, then $h < x^{\gamma} h^{1-\gamma}$ and

$$\begin{aligned}\omega(s_1 - f, x) &\leqslant 2\|s_1 - f\| \\ &\leqslant 2\omega(f, h) \\ &\leqslant c\omega(f, x^{\gamma} h^{1-\gamma})\end{aligned}$$

by (3.2). Therefore (3.3) follows.

Thus, we have obtained

THEOREM 3.1. *For the linear interpolating spline* s_1 *of* $f \in C(\Gamma)$, *there hold estimates* (3.1) *and* (3.3).

For $f \in H^{\mu}(\Gamma)$, we let $\nu = \gamma\mu$, then (3.3) becomes

$$\omega(s_1 - f, x) \leqslant cM_{\mu}(f) x^{\nu} h^{\mu-\nu}. \tag{3.3$'$}$$

Furthermore, we have

COROLLARY 3.2. *If* $f \in H^{\mu}(\Gamma)$, *then*

$$\|s_1(f) - f\|_{H^{\nu}} \leqslant cM_{\mu}(f) h^{\mu-\nu} \tag{3.4}$$

for $0 \leqslant \nu \leqslant \mu$.

Now we discuss the cubic interpolating splines. Here we require that

$$\frac{\max_j |\varDelta t_j|}{\min_j |\varDelta t_j|} \leqslant c_1 < \infty, \quad \max_{1 \leqslant j \leqslant N-1} \frac{|\varDelta t_j| + |\varDelta t_{j+1}|}{|\varDelta t_j + \varDelta t_{j+1}|} \leqslant c_2 < 2, \tag{3.5}$$

and the boundary conditions are given by

$$s_3'(t_0) = 0, \qquad s'(t_N) = 0, \quad \text{if } f \in C(\Gamma),$$

$$s_3'(t_0) = f'(t_0), \qquad s'(t_N) = f'(t_N), \quad \text{if } f \in C^1(\Gamma),$$

$$s_3''(t_0) = f''(t_0), \qquad s''(t_N) = f''(t_N), \quad \text{if } f \in C^2(\Gamma).$$

For $f \in C^m$, $m = 0, 1$ or 2, we have

$$\|s_3^{(r)} - f^{(r)}\| \leqslant c\omega(f^{(m)}, h) h^{m-r}, \qquad r = 0, 1, \ldots, m \tag{3.6}$$

and

$$\omega(s_3^{(m)}, x) \leqslant c\omega(f^{(m)}, x), \qquad x \geqslant 0. \tag{3.7}$$

Because of the similarity, we only give their proof under $m = 2$.

The cubic spline has the following expression:

$$\begin{aligned} s_3(t) =& \frac{M_{j-1}}{6\Delta t_j}(t_j - t)^3 + \frac{M_j}{6\Delta t_j}(t - t_{j-1})^3 \\ &+ \left(\frac{f_{j-1}}{\Delta t_j} - \frac{M_{j-1}\Delta t_j}{6}\right)(t_j - t) + \left(\frac{f_j}{\Delta t_j} - \frac{M_j \Delta t_j}{6}\right)(t - t_{j-1}), \\ & t \in \Gamma_j, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{3.8}$$

where $M_j = s_3''(t_j), j = 0, 1, \ldots, N$ which are determined by the equation

$$\mathrm{AM} = \mathrm{D} \tag{3.9}$$

with $\mathrm{M} = [M_0, M_1, \ldots, M_N]^T$. Here the matrixes A and D are given by

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{N-1} & 2 & \lambda_{N-1} \\ & & & 0 & 1 \end{bmatrix}$$

and

$$\mathrm{D} = [f_0'', 6f[t_0, t_1, t_2], \ldots, 6f[t_{N-2}, t_{N-1}, t_N], f_N'']^T,$$

respectively, where $\mu_j = \frac{\Delta t_j}{\Delta t_j + \Delta t_{j+1}}$, $\lambda_j = 1 - \mu_j$ and $f[t_{j-1}, t_j, t_{j+1}]$ is the second divided difference of f at the points t_{j-1}, t_j, t_{j+1}.

Let C^{N+1} be a $N + 1$ dimension complex space with maximum norm $||\cdot||$ and $\mathbf{x} = (x_0, x_1, \ldots, x_N)^T \in \mathrm{C}^{N+1}$. Suppose k such that $||\mathbf{x} = |x_k|$. We have

$$\begin{aligned} ||\mathrm{A}\mathbf{x}|| \geqslant& \begin{cases} |x_k| & \text{if } k = 0, N, \\ |\mu_k x_{k-1} + 2x_k + \lambda_k x_{k+1}| & \text{if } 1 \leqslant k \leqslant N - 1 \end{cases} \\ \geqslant& (2 - c_2)||\mathbf{x}||. \end{aligned}$$

It means

$$||\mathrm{A}^{-1}|| \leqslant 1/(2 - c_2). \tag{3.10}$$

Let $\mathbf{f}'' = [f_0'', f_1'', \ldots, f_N'']^T$. From

$$\begin{aligned}
&6f[t_{j-1}, t_j, t_{j+1}] - (\mu_j f_{j-1}'' + 2f_j'' + \lambda_j f_{j+1}'') \\
&\quad = 6(f[t_{j-1}, t_j, t_{j+1}] - \tfrac{1}{2}f_j'') + \mu_j(f_j'' - f_{j-1}'') + \lambda_j(f_j'' - f_{j+1}'')
\end{aligned}$$

and [1]

$$|f[t_{j-1}, t_j, t_{j+1}] - \tfrac{1}{2}f_j''| \leqslant c\omega(f'', |\Delta t_j|), \tag{3.11}$$

we have

$$||\mathrm{D} - \mathrm{A}\mathbf{f}''|| \leqslant c\omega(f'', h). \tag{3.12}$$

Then by (3.9), (3.10) and (3.12),

$$||\mathrm{M} - \mathbf{f}''|| = ||\mathrm{A}^{-1}(\mathrm{D} - \mathrm{A}\mathbf{f}'')|| \leqslant \frac{c}{2 - c_2}\omega(f'', h)$$

or

$$|M_j - f_j''| \leqslant c\omega(f'', h) \tag{3.13}$$

and also

$$|M_k - f''(t)| \leqslant c\omega(f'', h), \qquad k = j-1, \quad j, t \in \Gamma_j \tag{3.14}$$

for $j = 1, 2, \ldots, N$. From (3.8), the second derivative of the cubic spline is

$$s_3''(t) = \frac{t_j - t}{\Delta t_j}M_{j-1} + \frac{t - t_{j-1}}{\Delta t_j}M_j, \qquad t \in \Gamma_j, \quad j = 1, 2, \ldots, N \tag{3.15}$$

and from (3.14) and (3.15), we have

$$\begin{aligned}
|s_3''(t) - f''(t)| &= \left|\frac{t_j - t}{\Delta t_j}(M_{j-1} - f''(t)) + \frac{t - t_{j-1}}{\Delta t_j}(M_j - f''(t))\right| \\
&\leqslant \frac{|t_j - t| + |t - t_{j-1}|}{|\Delta t_j|}\max_{k=j-1,j}\ \{|M_k - f''(t)|\} \\
&\leqslant c\omega(f'', h), \qquad t \in \Gamma_j, \quad j = 1, 2, \ldots, N.
\end{aligned} \tag{3.16}$$

Thus, we have proved (3.6) for $r = 2$. If $r = 1$ or 0, (3.6) is easily obtained from the relations

$$s_3'(t_{j-1}) - f'(t_{j-1}) = \frac{1}{\Delta t_j}\int_{t_{j-1}}^{t_j}(t_j - \tau)[s_3''(\tau) - f''(\tau)]\,d\tau,$$

$$s_3'(t) - f'(t) = \int_{t_{j-1}}^{t}[s_3''(\tau) - f''(\tau)]\,d\tau + s_3'(t_{j-1}) - f'(t_{j-1}), \qquad t \in \Gamma_j,$$

and

$$s_3(t) - f(t) = \int_{t_{j-1}}^{t} [s_3'(\tau) - f'(\tau)]\, d\tau, \qquad t \in \Gamma_j,$$

where $j = 1, 2, \ldots, N$.

Let $t, t' \in \Gamma$. If $|t - t'| > h$, then, by (3.6),

$$\begin{aligned} |s_3''(t) - s_3''(t')| &\leqslant |s_3''(t) - f''(t)| + |f''(t) - f''(t')| + |f''(t') - s_3''(t')| \\ &\leqslant c'\omega(f'', h) + \omega(f'', |t - t'|) \\ &\leqslant c\omega(f'', |t - t'|), \end{aligned} \tag{3.17}$$

and if $0 < |t - t'| \leqslant h$, from

$$\begin{aligned} |s_3'''(t)| &= \frac{|M_j - M_{j-1}|}{|\Delta t_j|} \\ &\leqslant \frac{1}{|\Delta t_j|}(|M_j - f_j''| + |f_j'' - f_{j-1}''| + |f_{j-1}'' - M_{j-1}|) \\ &\leqslant c' \frac{\omega(f'', h)}{|\Delta t_j|} \\ &\leqslant c \frac{\omega(f'', h)}{h}, \qquad t \in \Gamma_j, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{3.18}$$

where we have used (3.13) and (3.5), it follows

$$\begin{aligned} |s_3''(t) - s_3''(t')| &\leqslant \left| \int_{t'}^{t} s_3'''(\tau)\, d\tau \right| \\ &\leqslant c' \frac{|t - t'|}{h} \omega(f'', h) \\ &\leqslant c' \frac{|t - t'|}{h} \left(1 + \frac{h}{|t - t'|}\right) \omega(f'', |t - t'|) \\ &\leqslant c\omega(f'', |t - t'|). \end{aligned} \tag{3.19}$$

Hence (3.7) is valid for $x > 0$. Of course, (3.7) is valid if $x = 0$, and so (3.7) is true. Similar to (3.3), we also have

$$\omega(s_3^{(m)} - f^{(m)}, x) \leqslant c\omega(f^{(m)}, x^{\gamma} h^{1-\gamma}), \qquad x \geqslant 0, \tag{3.20}$$

where $0 \leqslant \gamma \leqslant 1$. Thus we have proved

THEOREM 3.3. *For the cubic interpolating spline s_3 of $f \in C^m(\Gamma)$, there hold the estimates* (3.6) *and* (3.20) *where $m = 0, 1,$ or 2.*

For $f \in C^{m,\mu}(\Gamma)$, (3.20) becomes

$$\omega(s_3^{(m)} - f^{(m)}, x) \leqslant cM_\mu(f^{(m)})x^\nu h^{\mu-\nu}, \qquad 0 \leqslant \nu \leqslant \mu \tag{3.20$'$}$$

and thus we have

Corollary 3.4. *If $f \in C^{m,\mu}(\Gamma)$, then*

$$\|s_3(f) - f\|_{C^{m,\nu}} \leqslant cM_\mu(f^{(m)})h^{\mu-\nu}, \tag{3.21}$$

where $0 \leqslant \nu \leqslant \mu$.

4. COMPLEX SPLINE APPROXIMATION

In this section, we discuss the complex spline approximation for singular integral operators. For convenience, we assume that the operator A_w is bounded on $C^{m,\mu}(\Gamma)$ and let $s(\varphi)$ denote the linear or cubic interpolating spline of $\varphi \in C^{m,\mu}(\Gamma)$. Here the splines are those defined in the previous section and $m = 0, 1$, or 2. In the case of linear splines, m automatically equals 0.

It is effective to use complex splines for the numerical evaluation of Cauchy-type singular integrals. The following theorem is about their error estimates.

Theorem 4.1. *If $f \in C^{m,\mu}(\Gamma)$, then*

$$\|A_w s(f) - A_w(f)\|_{C^k} \leqslant cM_\mu(f^{(m)})h^{m+\mu-k-1}, \qquad 0 \leqslant k \leqslant m-1 \tag{4.1}$$

and

$$\|A_w s(f) - A_w(f)\|_{C^{m,\nu}} \leqslant c\|f\|_{C^{m,\mu}}h^{\mu-\nu}, \qquad 0 \leqslant \nu \leqslant \mu. \tag{4.2}$$

Proof. Using Theorem 2.4 and the results in Section 3, we have

$$\begin{aligned}\|A_w s(f) - A_w(f)\|_{C^k} &\leqslant c\|s(f) - (f)\|_{C^{k+1}} \\ &\leqslant cM_\mu(f^{(m)})h^{m+\mu-k-1}, \qquad 0 \leqslant k \leqslant m-1\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\|A_w s(f) - A_w(f)\|_{C^{m,\nu}} &\leqslant c\|s(f) - f\|_{C^{m,\nu}} \\ &\leqslant c\|f\|_{C^{m,\mu}}h^{\mu-\nu}, \qquad 0 \leqslant \nu \leqslant \mu.\end{aligned} \tag{4.4}$$

if $f \in C^{m,\mu}(\Gamma)$. The proof is complete. ∎

Now we consider the operator approximation by complex splines. Let

$$A_{w,\Delta} = \tilde{s}A_w s. \tag{4.5}$$

The spline operators $\tilde{s}$ and s may be different. For convenience, we assume $\tilde{s} = s$ here.

THEOREM 4.2. *If $\varphi \in C^{m,\mu}(\Gamma)$, then*

$$||A_{w,\Delta}(\varphi) - A_w(\varphi)||_{C^{m,v}} \leqslant c||\varphi||_{C^{m,\mu}} h^{\mu-v}, \qquad 0 \leqslant v \leqslant \mu. \tag{4.6}$$

Proof. By Corollary 3.2 or 3.4 and Theorem 2.4, we have

$$\begin{aligned} ||A_{w,\Delta}(\varphi) - A_w s(\varphi)||_{C^{m,v}} &\leqslant cM_\mu((A_w s(\varphi))^{(m)})h^{\mu-v} \\ &\leqslant c||s(\varphi)||_{C^{m,\mu}} h^{\mu-v} \\ &\leqslant c||\varphi||_{C^{m,\mu}} h^{\mu-v} \end{aligned} \tag{4.7}$$

and from Theorem 4.1, it follows that

$$\begin{aligned} &||A_{w,\Delta}(\varphi) - A_w(\varphi)||_{C^{m,v}} \\ &\quad \leqslant ||A_{w,\Delta}(\varphi) - A_w s(\varphi)||_{C^{m,v}} + ||A_w s(\varphi) - A_w(\varphi)||_{C^{m,v}} \\ &\quad \leqslant c||\varphi||_{C^{m,\mu}} h^{\mu-v}, \qquad 0 \leqslant v \leqslant \mu. \end{aligned} \tag{4.8}$$

The theorem has been proved. ∎

5. APPLICATIONS AND REMARKS

As a direct application of the results obtained in the previous section, we give a scheme for the numerical solution of singular integral equations over open arcs. We know that the Cauchy-type singular integral equation

$$a(t)\varphi(t) - \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau + \lambda\int_\Gamma k(t,\tau)\varphi(\tau)\,d\tau = f(t), \qquad t \in \Gamma \tag{5.1}$$

can be regularized as a Fredholm integral equation

$$(I + \lambda A_w K)y = f^*, \tag{5.2}$$

where $Ky(t) \equiv \int_\Gamma w_1(\tau)k(t,\tau)y(\tau)\,d\tau$, $y = \varphi/w_1$, $w_1 = [(a^2 - b^2)w]^{-1}$, $f^* = A_w f + bN_{\kappa-1}$ and $N_{\kappa-1}$ is a given polynomial with degree $\kappa - 1$ ($N_{\kappa-1} = 0$ if $\kappa \leqslant 0$) (for details, see [12]). Here all given functions are Hölder continuous,

λ is a constant, and undetermined function φ is restricted in h_0, which is a function class whose functions are integrable on Γ and are Hölder continuous on Γ except at the endpoints. Using $A_{w,\Delta}$ to replace A_w in (5.2), we have $(I+\lambda A_{w,\Delta}K)y_\Delta = f^*$, and then letting $u_\Delta = y_\Delta - f^*$, $g_\Delta = -\lambda A_{w,\Delta}Kf^*$, we get

$$(I+\lambda A_{w,\Delta}K)u_\Delta = g_\Delta. \tag{5.3}$$

If λ is not an eigenvalue of (5.2) and h is small enough, (5.3) has unique solution and the solution is a spline (cf. [9]). Thus, we can obtain the approximate solution of (5.1) by collocation method and the error estimate is easily obtained from Theorem 4.2. The problem will be discussed in detail in another paper.

For $\mu = 1$, the constants c and some other applications, we present some remarks.

Remark 1. The assumption $\mu < 1$ is only for convenience. Examining the proof of Lemma 2.2, we see that if $\mu = 1$ (2.10) is still true but (2.11) is not.

Remark 2. The constants c in theorems of Section 2 depend on a, b, m, μ and Γ. In Section 3, constants c depend on Γ and also depend on c_1 if the splines are cubic. Thus, in Theorem 4.1 and 4.2 the constants are related to a, b, m, μ, Γ and c_1, but do not depend on v.

Remark 3. Consider the following Cauchy-type integral:

$$Tf(t) = \int_\Gamma \frac{f(\tau)}{\tau - t}\,d\tau, \tag{5.4}$$

which is approximated by

$$T_\Delta f(t) = \int_\Gamma \frac{s(f)(\tau)}{\tau - t}\,d\tau. \tag{5.5}$$

If $g(\alpha) = g(\beta) = 0$, it is easy to prove

$$|Tg(t)| \leqslant c\int_0^1 \frac{\omega(g,y)}{y}\,dy, \qquad t\in\Gamma. \tag{5.6}$$

Thus, by (3.3) or (3.20) we have

$$|(T-T_\Delta)f(t)| \leqslant c\int_0^1 \frac{\omega(f-s(f),y)}{y}\,dy \leqslant \frac{c}{1-\gamma}\int_0^{h^\gamma} \frac{\omega(f,y)}{y}\,dy, \quad t\in\Gamma \tag{5.7}$$

where $0 \leqslant \gamma < 1$. This means that if f satisfies Dini condition then $T_\Delta f$ is uniformly convergent to Tf on Γ. We can also obtain

$$|(T - T_\Delta)f(t)| \leqslant \frac{c}{1-\gamma}\omega(f^{(m)}, h)h^{m-1+\gamma}, \qquad t \in \Gamma \tag{5.8}$$

if $f \in C^m$, $m = 1$ or 2.

Remark 4. If function w has integrable singularities at the endpoints of Γ, such as w_1 mentioned above, we can make a transformation with w multiplied by a simple polynomial $\sigma(t) = \beta - t, t - \alpha$, or $(\beta - t)(t - \alpha)$ such that the operator becomes a smooth operator divided by $\sigma(t)$ (cf. [16]). For example, the operator

$$T_1 f(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-\tau^2}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in (-1, 1) \tag{5.9}$$

can be transformed into

$$T_1 f(t) = \frac{1}{\sigma(t)}\left(\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\tau^2} f(\tau)}{\tau - t}\, d\tau + \frac{1}{\pi}\int_{-1}^{1} \frac{\tau + t}{\sqrt{1-\tau^2}} f(\tau)\, d\tau\right), \tag{5.10}$$

where $\sigma(t) = 1 - t^2$.

6. PROOF OF LEMMA 2.2

Proof. Equations (2.10) and (2.11) are equivalent to

$$|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)| \leqslant c M_\mu(f^{(m)})\delta^\mu(1 + |\ln \delta|) \tag{6.1}$$

or

$$|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)| \leqslant c\left(||u|| + \int_0^1 \frac{\omega(u, y)}{y}\, dy\right) M_\mu(f^{(m)})\delta^\mu, \tag{6.2}$$

if u satisfies Dini condition and $u(\alpha) = u(\beta) = 0$, where $t_1, t_2 \in \Gamma$ and $|t_2 - t_1| = \delta$. Now we prove (6.1) and (6.2).

For convenience, suppose $0 < \delta < 1$ and $\alpha \prec t_1 \prec t_2 \prec \beta$.

(i) If $|t_1 - \alpha| > \delta$ and $|\beta - t_2| \leqslant \delta$, let

$$\begin{aligned}
&|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)| \\
&\quad = m!\left|\left(\int_{\widehat{\alpha t_1}} + \int_{\widehat{t_1 \beta}}\right) u(\tau)\left[\frac{f(\tau) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} - \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_1)^{m+1}}\right] d\tau\right| \\
&\quad = |I_1 + I_2|.
\end{aligned}$$

From (2.5) *and* $|\widehat{t_1\beta}| = |\widehat{t_1t_2}| + |\widehat{t_2\beta}| \leqslant 2c_0\delta$, it follows that

$$\begin{aligned}|I_2| &= m!\left|\int_{\widehat{t_1\beta}} u(\tau)\left[\frac{f(\tau)-T_m(f)(\tau,t_2)}{(\tau-t_2)^{m+1}} - \frac{f(\tau)-T_m(f)(\tau,t_1)}{(\tau-t_1)^{m+1}}\right]d\tau\right| \\ &\leqslant c||u||M_\mu(f^{(m)})\int_{\widehat{t_1\beta}}(|\tau-t_2|^{\mu-1}+|\tau-t_1|^{\mu-1}|)|\,d\tau|. \\ &\leqslant c||u||M_\mu(f^{(m)})\delta^\mu. \end{aligned} \tag{6.3}$$

Define the function

$$h(\tau) = T_m(f)(\tau,t_1) - T_m(f)(\tau,t_2), \qquad \tau\in\Gamma. \tag{6.4}$$

Then

$$h(\tau) = h(t_2) + h'(t_2)(\tau-t_2) + \cdots + \frac{h^{(m)}(t_2)}{m!}(\tau-t_2)^m \tag{6.5}$$

because $h(\tau)$ is an m-degree polynomial of τ. Noting that

$$h^{(k)}(t_2) = T_{m-k}(f^{(k)})(t_2,t_1) - f^{(k)}(t_2) \tag{6.6}$$

and (2.5), we have

$$\frac{m!}{k!}|h^{(k)}(t_2)| \leqslant c\binom{m}{k}M_\mu(f^{(m)})\delta^{m+\mu-k} \tag{6.7}$$

for $k = 1,2,\ldots,m$, therefore

$$\begin{aligned} i_1 &\equiv m!\left|\int_{\widehat{t_1\beta}} u(\tau)\frac{T_m(f)(\tau,t_2)-T_m(f)(\tau,t_1)}{(\tau-t_2)^{m+1}}d\tau\right| \\ &\leqslant m!\sum_{k=0}^{m}\frac{|h^{(k)}(t_2)|}{k!}\int_{\widehat{\alpha t_1}}\frac{|u(\tau)|}{|\tau-t_2|^{m-k+1}}|d\tau| \\ &\leqslant cM_\mu(f^{(m)})\sum_{k=0}^{m}\binom{m}{k}\delta^{m-k+\mu}\int_{\widehat{\alpha t_1}}\frac{|u(\tau)|}{|\tau-t_2|^{m-k+1}}|d\tau| \\ &\leqslant cM_\mu(f^{(m)})||u||\delta^\mu\left(1+\ln\frac{1}{\delta}\right). \end{aligned} \tag{6.8}$$

If u satisfies Dini condition and $u(\beta) = 0$,

$$\int_{\widehat{\alpha t_1}} \frac{|u(\tau)|}{|\tau - t_2|}|d\tau| = \int_{\widehat{\alpha t_1}} \frac{|u(\tau) - u(\beta)|}{|\tau - t_2|}|d\tau| \leqslant \int_{\widehat{\alpha t_1}} \frac{\omega(u, |\tau - \beta|)}{|\tau - t_2|}|d\tau|.$$

For $\tau \in \widehat{\alpha t_1}$,

$$c_0|\tau - t_2| \geqslant |\widehat{\tau t_2}| = |\widehat{\tau t_1}| + |\widehat{t_1 t_2}| \geqslant |t_2 - t_1| = \delta, \tag{6.9}$$

but $|\beta - t_2| \leqslant \delta$, thus $|\beta - t_2| \leqslant c_0|\tau - t_2|$ and

$$|\tau - \beta| \leqslant |\tau - t_2| + |t_2 - \beta| \leqslant (1 + c_0)|\tau - t_2|. \tag{6.10}$$

So we have

$$\int_{\widehat{\alpha t_1}} \frac{|u(\tau)|}{|\tau - t_2|}|d\tau| \leqslant \int_{\widehat{\alpha t_1}} \frac{\omega(u, (1 + c_0)|\tau - t_2|)}{|\tau - t_2|}|d\tau| \leqslant c\int_0^1 \frac{\omega(u, y)}{y}\, dy \tag{6.11}$$

and

$$i_1 \leqslant cM_\mu(f^{(m)})\left(||u|| + \int_0^1 \frac{\omega(u, y)}{y}\, dy\right)\delta^\mu. \tag{6.12}$$

By (2.5),

$$\begin{aligned} i_2 &= m!\left|\int_{\widehat{\alpha t_1}} u(\tau)[f(\tau) - T_m(f)(\tau, t_1)]\left[\frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}}\right]d\tau\right| \\ &\leqslant c||u||M_\mu(f^{(m)})\int_{\widehat{\alpha t_1}} \frac{|\tau - t_1|^{m+\mu}|(\tau - t_1)^{m+1} - (\tau - t_2)^{m+1}|}{|\tau - t_1|^{m+1}|\tau - t_2|^{m+1}}|d\tau| \\ &\leqslant c||u||M_\mu(f^{(m)})|t_1 - t_2| \sum_{k=0}^{m} \int_{\widehat{\alpha t_1}} \frac{|\tau - t_1|^{k+\mu-1}}{|\tau - t_2|^{k+1}}|d\tau|. \end{aligned} \tag{6.13}$$

Let $l_1 = |\widehat{\alpha t_1}|$, $l_2 = |\widehat{\alpha t_2}|$ and s be the parameter of arc length. Using integration by parts and $l_1 < l_2$,

$$\begin{aligned} \int_{\widehat{\alpha t_1}} \frac{|\tau - t_1|^{k+\mu-1}}{|\tau - t_2|^{k+1}}|d\tau| &\leqslant c\int_0^{l_1} \frac{|\widehat{\tau t_1}|^{k+\mu-1}}{|\widehat{\tau t_2}|^{k+1}}\, ds = c\int_0^{l_1} \frac{(l_1 - s)^{k+\mu-1}}{(l_2 - s)^{k+1}}\, ds \\ &\leqslant c\left[\frac{k+1}{k+\mu}\int_0^{l_1} \frac{(l_1 - s)^{k+\mu}}{(l_2 - s)^{k+2}}\, ds - \frac{1}{k+\mu}\frac{l_1^{k+\mu}}{l_2^{k+1}}\right] \end{aligned}$$

$$\leqslant c\frac{k+1}{k+\mu}\int_0^{l_1}\frac{(l_2-s)^{k+\mu}}{(l_2-s)^{k+2}}\,ds$$

$$\leqslant c\frac{k+1}{k+\mu}\frac{1}{1-\mu}\left[\frac{1}{(l_2-l_1)^{1-\mu}}-\frac{1}{l_2^{1-\mu}}\right],$$

but $l_2-l_1=|\widehat{t_1t_2}|>\delta$, then

$$\int_{\widehat{\alpha t_1}}\frac{|\tau-t_1|^{k+\mu-1}}{|\tau-t_2|^{k+1}}|d\tau|\leqslant c'\frac{k+1}{k+\mu}\frac{1}{1-\mu}\frac{1}{(l_2-l_1)^{1-\mu}}\leqslant c\delta^{\mu-1} \tag{6.14}$$

for $k=0,1,\ldots,m$, so we obtain

$$i_2\leqslant cM_\mu(f^{(m)})\delta^\mu. \tag{6.15}$$

Thus from

$$\begin{aligned}I_1 &= m!\int_{\widehat{\alpha t_1}}u(\tau)\left[\frac{f(\tau)-T_m(f)(\tau,t_2)}{(\tau-t_2)^{m+1}}-\frac{f(\tau)-T_m(f)(\tau,t_1)}{(\tau-t_1)^{m+1}}\right]d\tau\\ &= m!\int_{\widehat{\alpha t_1}}u(\tau)\left\{\frac{T_m(f)(\tau,t_1)-T_m(f)(\tau,t_2)}{(\tau-t_2)^{m+1}}\right.\\ &\quad\left.+[f(\tau)-T_m(f)(\tau,t_1)]\left(\frac{1}{(\tau-t_2)^{m+1}}-\frac{1}{(\tau-t_1)^{m+1}}\right)\right\}d\tau,\end{aligned} \tag{6.16}$$

$$\begin{aligned}|I_1|&\leqslant i_1+i_2\\ &\leqslant\begin{cases}cM_\mu(f^{(m)})\|u\|\delta^\mu(1+|\ln\delta|) & \text{if } u(\beta)\neq 0,\\ cM_\mu(f^{(m)})\left(\|u\|+\int_0^1\frac{\omega(u,y)}{y}dy\right)\delta^\mu & \text{if } u(\beta)=0\end{cases}\end{aligned} \tag{6.17}$$

and together with (6.3), we obtain

$$\begin{aligned}|S_{u,m}(f)(t_1)-S_{u,m}(f)(t_2)|&\leqslant|I_1|+|I_2|\\ &\leqslant\begin{cases}cM_\mu(f^{(m)})\|u\|\delta^\mu(1+|\ln\delta|) & \text{if } u(\beta)\neq 0,\\ cM_\mu(f^{(m)})\left(\|u\|+\int_0^1\frac{\omega(u,y)}{y}dy\right)\delta^\mu & \text{if } u(\beta)=0\end{cases}\end{aligned} \tag{6.18}$$

and there follows (6.1) and (6.2).

(ii) If $|t_1-\alpha|\leqslant\delta$ and $|\beta-t_2|>\delta$, similarly, we also have (6.18) but α in place of β. Hence, (6.1) and (6.2) are valid.

(iii) If $|t_1-\alpha|>\delta$ and $|\beta-t_2|>\delta$, let

$$\begin{aligned}
&|S_{u,m}(f)(t_1)-S_{u,m}(f)(t_2)|\\
&= m!\left|\left(\int_{\widehat{\alpha t_1}}+\int_{\widehat{t_1t_2}}+\int_{\widehat{t_2\beta}}\right)u(\tau)\left[\frac{f(\tau)-T_m(f)(\tau,t_2)}{(\tau-t_2)^{m+1}}\right.\right.\\
&\quad\left.\left.-\frac{f(\tau)-T_m(f)(\tau,t_1)}{(\tau-t_1)^{m+1}}\right]d\tau\right|\\
&= |I_1+I_2+I_3|. \qquad (6.19)
\end{aligned}$$

Similar to the proof of (6.3),

$$|I_2|\leqslant cM_\mu(f^{(m)})\delta^\mu. \qquad (6.20)$$

We rewrite I_1+I_3 as i_1+i_2 where

$$\begin{aligned}
i_1 =&\, m!\int_{\widehat{\alpha t_1}}u(\tau)\frac{T_m(f)(\tau,t_1)-T_m(f)(\tau,t_2)}{(\tau-t_2)^{m+1}}d\tau\\
&+m!\int_{\widehat{t_2\beta}}u(\tau)\frac{T_m(f)(\tau,t_1)-T_m(f)(\tau,t_2)}{(\tau-t_1)^{m+1}}d\tau, \qquad (6.21)
\end{aligned}$$

$$\begin{aligned}
i_2 =&\, m!\int_{\widehat{\alpha t_1}}u(\tau)[f(\tau)-T_m(f)(\tau,t_1)]\left(\frac{1}{(\tau-t_2)^{m+1}}-\frac{1}{(\tau-t_1)^{m+1}}\right)d\tau\\
&+ m!\int_{\widehat{t_2\beta}}u(\tau)[f(\tau)-T_m(f)(\tau,t_2)]\left(\frac{1}{(\tau-t_2)^{m+1}}-\frac{1}{(\tau-t_1)^{m+1}}\right)d\tau. \qquad (6.22)
\end{aligned}$$

Using (6.4) and (6.5), we also rewrite i_1 as

$$\begin{aligned}
&m!\sum_{k=0}^{m-1}\left[\frac{h^{(k)}(t_2)}{k!}\int_{\widehat{\alpha t_1}}\frac{u(\tau)}{(\tau-t_2)^{m-k+1}}d\tau+\frac{h^{(k)}(t_1)}{k!}\int_{\widehat{t_2\beta}}\frac{u(\tau)}{(\tau-t_1)^{m-k+1}}d\tau\right]\\
&+\left(\int_{\widehat{\alpha t_1}}\frac{u(\tau)}{\tau-t_2}d\tau+\int_{\widehat{t_2\beta}}\frac{u(\tau)}{\tau-t_1}d\tau\right)[f^{(m)}(t_1)-f^{(m)}(t_2)]\\
&= i_{11}+i_{12}. \qquad (6.23)
\end{aligned}$$

Similar to the proof of (6.8) and (6.15), we have

$$|i_{11}| \leqslant c||u||M_\mu(f^{(m)})\delta^\mu, \tag{6.24}$$

$$|i_{12}| \leqslant c||u||M_\mu(f^{(m)})\delta^\mu\left(1+\ln\frac{1}{\delta}\right) \tag{6.25}$$

and

$$|i_2| \leqslant c||u||M_\mu(f^{(m)})\delta^\mu. \tag{6.26}$$

If u satisfies Dini condition and $u(\alpha)=u(\beta)=0$, we have

$$\begin{aligned}
&\left|\int_{\widehat{\alpha t_1}}\frac{u(\tau)}{\tau-t_2}d\tau+\int_{\widehat{t_2\beta}}\frac{u(\tau)}{\tau-t_1}d\tau\right| \\
&=\left|\int_{\widehat{\alpha t_1}}\frac{u(\tau)-u(t_2)}{\tau-t_2}d\tau+\int_{\widehat{t_2\beta}}\frac{u(\tau)-u(t_1)}{\tau-t_1}d\tau\right. \\
&\quad\left.+u(t_2)\int_{\widehat{\alpha t_1}}\frac{1}{\tau-t_2}d\tau+u(t_1)\int_{\widehat{t_2\beta}}\frac{1}{\tau-t_1}d\tau\right| \\
&\leqslant\int_{\widehat{\alpha t_1}}\frac{\omega(u,|\tau-t_2|)}{|\tau-t_2|}|d\tau|+\int_{\widehat{t_2\beta}}\frac{\omega(u,|\tau-t_1|)}{\tau-t_1}|d\tau| \\
&\quad+\left|u(t_2)\left[\ln\left|\frac{t_1-t_2}{\alpha-t_2}\right|+i\theta_1\right]+u(t_1)\left[\ln\left|\frac{\beta-t_1}{t_1-t_2}\right|+i\theta_2\right]\right|,
\end{aligned} \tag{6.27}$$

where θ_1 is the angle between t_1-t_2 and $\alpha-t_2$ and θ_2 is the angle between $\beta-t_1$ and t_2-t_1, and

$$\begin{aligned}
&\left|u(t_2)\left[\ln\left|\frac{t_1-t_2}{\alpha-t_2}\right|+i\theta_1\right]+u(t_1)\left[\ln\left|\frac{\beta-t_1}{t_1-t_2}\right|+i\theta_2\right]\right| \\
&=|[u(t_2)-u(t_1)]\ln|t_2-t_1|-[u(t_2)-u(\alpha)]\ln|\alpha-t_2| \\
&\quad+[u(t_1)-u(\beta)]\ln|\beta-t_1|+i[\theta_1 u(t_2)+\theta_2 u(t_1)]| \\
&\leqslant c\left[||u||+\sup_{0<x\leqslant 1}\omega(u,x)\ln\frac{1}{x}\right],
\end{aligned} \tag{6.28}$$

therefore

$$\left|\int_{\widehat{\alpha t_1}} \frac{u(\tau)}{\tau - t_2} d\tau + \int_{\widehat{t_2\beta}} \frac{u(\tau)}{\tau - t_1} d\tau\right| \leqslant c\left(\|u\| + \int_0^1 \frac{\omega(u,y)}{y} dy\right), \tag{6.29}$$

so that

$$|i_{12}| \leqslant cM_\mu(f^{(m)})\left(\|u\| + \int_0^1 \frac{\omega(u,y)}{y} dy\right)\delta^\mu. \tag{6.30}$$

Now we obtain

$$\begin{aligned} |i_1| &\leqslant |i_{11}| + |i_{12}| \\ &\leqslant \begin{cases} cM_\mu(f^{(m)})\left(\|u\| + \int_0^1 \dfrac{\omega(u,y)}{y} dy\right)\delta^\mu & \text{if } u(\alpha) = u(\beta) = 0, \\ cM_\mu(f^{(m)})\|u\|\delta^\mu\left(1 + \ln\dfrac{1}{\delta}\right) & \text{otherwise} \end{cases} \end{aligned} \tag{6.31}$$

and together with (6.20) and (6.26), (6.1) and (6.2) are valid in this case.

(iv) If $|t_1 - \alpha| \leqslant \delta$ and $|\beta - t_2| \leqslant \delta$, then $|\Gamma| = |\widehat{\alpha t_1}| + |\widehat{t_1 t_2}| + |\widehat{t_2\beta}| \leqslant 3c_0\delta$. From (2.5),

$$\begin{aligned} &|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)| \\ &\quad \leqslant c\|u\|M_\mu(f^{(m)})\int_\Gamma (|\tau - t_1|^{-1+\mu} + |\tau - t_2|^{-1+\mu})|d\tau| \\ &\quad \leqslant c\|u\|M_\mu(f^{(m)})|\Gamma|^\mu \\ &\quad \leqslant 3c_0c\|u\|M_\mu(f^{(m)})\delta^\mu \end{aligned} \tag{6.32}$$

and then (6.1) and (6.2) are valid.

Now we have proved that (6.1) and (6.2) are true in each case. The proof is complete. ▮

ACKNOWLEDGMENTS

The author is grateful to the editor and the reviewers for their valuable suggestions. The author also thanks Professor Wei Lin, Dr. John H Easton and Dr. Xiao Bin Liu for their generous support and assistance.

REFERENCES

1. J. H. Ahlberg, E. N. Nilson, and J. L. Wash, Complex cubic splines, *Trans. Amer. Math. Soc.* **129** (1967), 391–418.

2. J. H. Ahlberg, E. N. Nilson, and J. L. Wash, Properties of analytic splines. I. Complex polynomial splines, *J. Math. Anal. Appl.* **27** (1969), 262–278.
3. D. Berthold and D. Junghanns, New error bounds for the quadrature method for the solution of Cauchy singular integral equations, *SIAM J. Numer. Anal.* **30** (1993), 1351–1372.
4. H. L., Cheng, On complex splines, *Sci. Sinica* **24** (1981), 160–169.
5. J. Y. Du, Some systems of orthogonal polynomials associated with singular integral equations, *Acta Math. Sci.* **7** (1987), 85–96.
6. D. Elliott, Orthogonal polynomials associated with singular integral equations having a Cauchy kernel, *SIAM J. Math. Anal.* **13** (1982), 1041–1052.
7. D. Elliott, Rates of convergence for the method of classical collocation method for solving singular integral equations, *SIAM J. Numer. Anal.* **21** (1984), 136–148.
8. X. L. Huang, Approximate solutions to singular integral equations, *Acta Math. Sci.* **12** (1992), 1, 75–85.
9. R. Kress, "Linear Integral Equation," Springer-Verlag, Berlin, 1989.
10. C. K. Lu, Error analysis for interpolating complex cubic splines with deficiency 2, *J. Approx. Theory* **36** (1982), 183–196.
11. C. K. Lu, The approximation of Cauchy-type integrals by some kinds of interpolatory splines, *J. Approx. Theory* **36** (1982), 197–212.
12. C. K. Lu, "Boundary Value Problems for Analytic Functions," World Scientific, Singapore, 1993.
13. J. K. Lu and J. Y. Du, Numerical methods of singular integral equations, *Advanced Math.* **23** (1994), 424–431.
14. S. G. Mikhlin and S. Prössdorf, "Singular Integral Operators," Springer-Verlag, Berlin, 1986.
15. N. I. Mushelishvili, "Singular Integral Equations," Noordhoff, Gröningen, 1962.
16. Y. J. Xu, On stability of negative index singular integral equations over an interval, *Integral Equations Operator Theory*, to appear.
17. Y. J. Xu, A kind of stability of solution of singular integral equations, *Acta. Math. Sci.* **11** (1991), 448–456.
18. A. Zygmund, "Trigonometric Series," Cambridge Univ. Press, Cambridge, UK 1968.